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SIMULATION OF COMBINED SYSTEMS BY PERIODIC STRUCTURES: THE WAVE TRANSFER MATRIX APPROACH

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An exact closed-form method is presented for frequency domain analysis of linear uniform combined systems. The proposed method is based on the idea that such systems can be treated as if they were periodic structures under multiple excitations. In other words, the continuous system is viewed as subdivided into small equispaced subsystems so that all the arbitrarily located point-wise discontinuities (i.e., external and boundary disturbances, forces exerted by constraints and attached discrete systems) appear as acting at subsystem interfaces. By adapting the periodic structure wave solution, the response of the combined system is found to be formed by a *free wave field* incorporating the dynamics of the entire system and a *forced wave field* generated by discontinuities in both directions, as if the system were infinite in extent. In order to validate the theory, two examples are considered. In the first example, the phase closure principle is invoked to predict the free and forced motion of a translating string constrained by arbitrarily spaced linear springs. In the second example, formulas for natural frequencies of beams on multiple constraint supports with different boundary conditions are obtained from those of beams with simply supported ends.

1. INTRODUCTION

Many engineering structures can be modelled as an assemblage of one-dimensional continuous systems arbitrarily combined with discrete elements and supports. The dynamic behaviour of such systems has been extensively investigated by many authors using various techniques. Without any attempt at a complete survey, there are approaches such as the Lagrange multiplier method by Dowell [1, 2], the receptance method by Bishop and Johnson [3], the transfer matrix method by Pestel and Leckie [4], the Green's function method by Bergman and co-authors [5, 6] and Kelkel [7], and recently the transfer function method by Yang [8] and Tan and Chung [9].

The object of the present paper is to show how frequency domain analysis of combined/constrained systems can be performed by adapting the multi-excited periodic structures wave solution, a concept developed by the first author in his recent dissertation [10]. A structure is termed periodic if it is composed of spatially repeated elements coupled in identical ways. The basic idea behind the theory described in this work is that any continuous system can be seen as being arbitrarily subdivided into small subsystems of equal length in such a way that all the external and constraint forces coincide at numbers of such *imaginary* subsystem interfaces. It should be mentioned that a similar concept was used by Engels and Meirovitch [11] although their study was limited to (the case of) single span beams under distributed loads. On the other hand, periodic structure wave analysis

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methods, which fully exploit their spatial properties, may be interpreted as based on simulating continuous systems. This is because both systems exhibit similar behavior in terms of wave propagation. This similarity has rendered the analysis of periodic structures computationally simple. In other words, irrespective of the dimensions and degrees of complexity of these structures, their global dynamic behavior, as in continuous systems, is captured once their corresponding dispersion relation has been determined, which can be derived from the single periodic element dynamics alone in conjunction with the boundary conditions. This has enabled several authors to develop basically similar methods that are applicable to both periodic and continuous system networks. Among these, one would mention the wave transfer matrix by von Flotow [12], the phase closure method by Signorelli and von Flotow [13], the wave receptance analysis developed by Mead [14] for periodic structures and extended by Mead and Yaman [15] to continuous beams on multiple supports, the "decaying wave method" presented by Yong and Lin [16] for piecewise periodic truss beams and subsequently extended by Cai and Lin [17] to include frame structures.

The procedure of analysis presented here begins by describing the dynamics of combined systems using a discrete equation of motion borrowed from the multi-excited periodic structure solution algorithm developed by Meirovitch and Engels [18]. A wave perspective is then incorporated into the solution in order to simplify the computational effort and to gain a physical insight into the dynamics of these systems. The formal response of combined systems is found to be formed by two parts: a globally circumnavigating *free wave field*, which contains the implicit contribution of the detailed dynamics of the entire system, expressible in terms of one of the system's extreme ends incoming wave vector; and a *forced wave field* consisting of the direct contribution of constraint forces, which is therefore independent of the system's extreme boundary conditions. These two wave fields are determined by accounting the reflection process at the extreme ends of the system in conjunction with force constraint conditions at all points of discontinuity along the combined systems.

In order to verify and assess the accuracy of the proposed method two example systems are considered in which numerical results are given and compared with those quoted by other investigators. In the first example, the phase closure principle is invoked to determine the natural frequencies and mode shapes of a translating string constrained by arbitrarily spaced linear springs. In the second example, natural frequency formulas of multi-constrained beams with simply supported ends (a simpler end condition) are used to derive those of beams with other extreme boundary conditions.

Wave propagation is central to the method presented in this paper; therefore, for the sake of clarity, the subsequent section offers a brief review of wave propagation description of continuous system dynamics.

2. WAVE DYNAMICS OF CONTINUOUS SYSTEMS

2.1. DISPERSION RELATION AND WAVE VECTORS

The motion of a continuous uniform segment of a one-dimensional system, vibrating with frequency ω, can be described by a $2n$ first order ordinary state space equation as

$$(\partial/\partial x)\boldsymbol{\eta}(x,\omega) = \mathbf{F}(\omega)\boldsymbol{\eta}(x,\omega) \tag{1}$$

where the state vector $\boldsymbol{\eta}(x,\omega)$ may be given in terms of cross-sectional deflection variables $\mathbf{d}(x,\omega)$ and their spatial derivatives, or forces $\mathbf{f}(x,\omega)$, each of order n. The state matrix $\mathbf{F}(\omega)$ may incorporate the distributed mechanical properties of a general damped non-self-adjoint continuous model. Since all these quantities are assumed to be frequency dependent, the time factor $e^{j\omega t}$ is omitted in (1) and in the remainder of this paper.

The solution of (1) defines the relations of the sectional mechanical state vector at an arbitrary point x by means of that at reference point x_0 as

$$\boldsymbol{\eta}(x, \omega) = e^{F(x-x_0)}\boldsymbol{\eta}(x_0, \omega) = \mathbf{A}(x, x_0, \omega)\boldsymbol{\eta}(x_0. \omega), \tag{2}$$

where matrix $\mathbf{A}$ is the transfer matrix. Following von Flotow [12], the wave propagation characteristic of the segment is revealed by introducing the transformation

$$\boldsymbol{\eta}(x, \omega) = \mathbf{P}(\omega)\mathbf{v}(x, \omega), \tag{3}$$

where columns in $\mathbf{P}(\omega)$ are the eigenvectors of $\mathbf{F}(\omega)$ and the wave state vector $\mathbf{v}(x, \omega)$ may be given as an arrangement of the positive-going $\mathbf{v}^+(x, \omega)$ and negative-going $\mathbf{v}^-(x, \omega)$ wave components of any of the cross-sectional physical variables. Then, the equivalent of (2) in wave state description is

$$\begin{Bmatrix} \mathbf{v}^+(x, \omega) \\ \mathbf{v}^-(x, \omega) \end{Bmatrix} = [\mathbf{T}(x, x_0, \omega)]\begin{Bmatrix} \mathbf{v}^+(x_0, \omega) \\ \mathbf{v}^-(x_0, \omega) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^+(x, x_0, \omega) & \\ & \boldsymbol{\Lambda}^-(x, x_0, \omega) \end{bmatrix}\begin{Bmatrix} \mathbf{v}^+(x_0, \omega) \\ \mathbf{v}^-(x_0, \omega) \end{Bmatrix}, \tag{4}$$

where the diagonal matrix $\mathbf{T}$ is the wave transfer matrix and

$$\boldsymbol{\Lambda}^+(x, x_0, \omega) = \begin{bmatrix} e^{\gamma_1(x-x_0)} & & \\ & \ddots & \\ & & e^{\gamma_n(x-x_0)} \end{bmatrix}, \qquad \boldsymbol{\Lambda}^-(x, x_0, \omega) = \begin{bmatrix} e^{\gamma_{n+1}(x-x_0)} & & \\ & \ddots & \\ & & e^{\gamma_{2n}(x-x_0)} \end{bmatrix} \tag{5}$$

The distinct eigenvalues $\gamma_j(\omega)$ of $\mathbf{F}(\omega)$ are known as propagation constants, which are generally complex. Their imaginary parts (wave numbers) indicate the phase variation and their real parts (attenuation coefficients) represent the rate of decay of the associated wave components. Generally, the eigenvalues of stationary structural models occur in pairs $(-\gamma_i = \gamma_{i+n}, i = 1, \ldots, n)$ implying that the positive-going and their negative-going wave counterparts travel with the same phase. This property is lost in gyroscopic systems such as axially moving and rotating models.

2.2. WAVE INTERPRETATION OF RESONANCE: PHASE CLOSURE

Now suppose that the continuous system's left and right boundaries are located at points x_0 and x_m, respectively. With reference to Figure 1, wave states at the system boundaries are described by the following equations

$$\mathbf{v}^-(x_0, \omega) = \boldsymbol{\Lambda}^-(x_0, x_m, \omega)\mathbf{v}^-(x_m, \omega), \qquad \mathbf{v}^-(x_m, \omega) = \mathbf{R}(x_m)\mathbf{v}^+(x_m, \omega),$$
$$\mathbf{v}^+(x_m, \omega) = \boldsymbol{\Lambda}^+(x_m, x_0, \omega)\mathbf{v}^+(x_0, \omega) \qquad \mathbf{v}^+(x_0, \omega) = \mathbf{R}(x_0)\mathbf{v}^-(x_0, \omega), \tag{6–9}$$

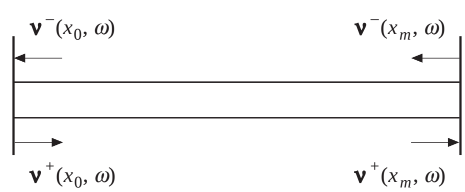

Figure 1. Wave motion along a finite continuous system.

where $\mathbf{R}$ is the reflection matrix and may be frequency dependent. This matrix is formed by transforming the system's homogeneous boundary conditions in terms of state vectors $\boldsymbol{\eta}$,

$$\mathbf{B}_i \boldsymbol{\eta}(x_i, \omega)|_{i=0,m} = \mathbf{0}, \tag{10}$$

into wave vectors $\mathbf{v}$,

$$[\mathbf{B}_i][\mathbf{P}(\omega)]\begin{Bmatrix}\mathbf{v}^+(x_i, \omega)\\ \mathbf{v}^-(x_i, \omega)\end{Bmatrix}\Bigg|_{i=0,m} = \mathbf{0}, \tag{11}$$

where $\mathbf{B}$ is the frequency domain equivalent of the time domain boundary operators matrix. Hence, the system ends reflection matrices are determined by comparing equations (7) and (9) with equations (11).

The phase closure principle [19] states that natural frequencies occur when all wave components, after multiple reflection at all constraints along its path, complete a circumnavigation of the entire system maintaining the same phase and amplitude. Hence, for the continuous system in consideration, resonance will occur provided that the circumnavigating wave vector $\mathbf{v}^-(x_0, \omega)$ satisfies the following relation

$$[\boldsymbol{\Lambda}^-(x_0, x_m, \omega)\mathbf{R}(x_m)\boldsymbol{\Lambda}^+(x_m, x_0, \omega)\mathbf{R}(x_0) - \mathbf{I}]\mathbf{v}^-(x_0, \omega) = \mathbf{0}, \tag{12}$$

where $\mathbf{I}$ is a unitary matrix. Thus,

$$\det[\boldsymbol{\Lambda}^-(x_0, x_m, \omega)\mathbf{R}(x_m)\boldsymbol{\Lambda}^+(x_m, x_0, \omega)\mathbf{R}(x_0) - \mathbf{I}] \equiv \det \boldsymbol{\Delta}_0 = 0 \tag{13}$$

gives the continuous system natural frequencies. From equations (3), (4) and (9), the corresponding mode shapes are given by

$$\begin{aligned}\mathbf{d}(x, \omega) &= \mathbf{P}_{11}\mathbf{v}^+(x, \omega) + \mathbf{P}_{12}\mathbf{v}^-(x, \omega)\\ &= [\mathbf{P}_{11}\boldsymbol{\Lambda}^+(x, x_0, \omega)\mathbf{R}(x_0) + \mathbf{P}_{12}\boldsymbol{\Lambda}^-(x, x_0, \omega)]\mathbf{v}^-(x_0, \omega).\end{aligned} \tag{14}$$

Similar derivations of such solutions are reported in [13, 20] for periodic truss beams.

3. WAVE DYNAMICS OF SYSTEMS UNDER POINT-WISE DISCONTINUITIES

3.1. RESPONSE OF PERIODIC STRUCTURES

It is well known that the dynamics of a periodic structure under multiple excitations can be described in terms of any two neighbouring periodic elements [18]

$$\boldsymbol{\eta}(x_{i+1}, \omega) = \mathbf{A}(x_{i+1}, x_i, \omega)\boldsymbol{\eta}(x_i, \omega) + \boldsymbol{\eta}^*(x_{i+1}, \omega), \tag{15}$$

where $\boldsymbol{\eta}(x_i, \omega)$ denotes the *modified mechanical state*, i.e., it contains the internal state plus the applied disturbances at, say the left end of the periodic element i, $\mathbf{A}(x_{i+1}, x_i, \omega)$ is the transfer matrix of the generic periodic element, and $\boldsymbol{\eta}^*(x_{i+1}, \omega)$ is the *net* applied disturbance between elements i and $i+1$.

The solution of equation (15) gives the response of the periodic structure in the form

$$\boldsymbol{\eta}(x_i, \omega) = \mathbf{A}(x_i, x_0. \omega)\boldsymbol{\eta}(x_0, \omega) + \sum_{k=0}^{i-1} \mathbf{A}(x_i, x_k, \omega)\boldsymbol{\eta}^*(x_k, \omega). \tag{16}$$

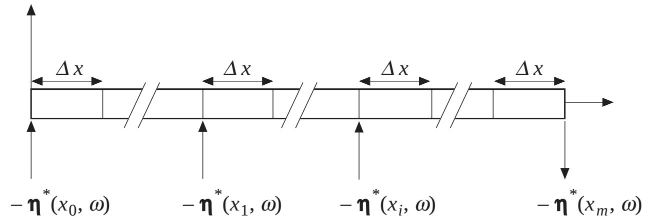

Figure 2. Schematic of a continuous system under arbitrarily spaced pointwise discontinuities viewed as composed of repeated continuous subsystems of length Δx.

3.2. THE SIMULATION CONCEPT

Consider a continuous one-dimensional system under arbitrarily located point-wise discontinuities $\boldsymbol{\eta}^*(x_i, \omega)$; $i = 0, 1, \ldots, m$; as shown in Figure 2. Here, discontinuities include: applied and boundary disturbances and forces exerted by supports or discrete systems, $\mathbf{f}^*(x_i, \omega)$, and imposed deflections $\mathbf{d}^*(x_i, \omega)$. All these discontinuities, although arbitrarily distanced, appear as acting at numbers of the repeated continuous subsystems of equal length Δx. Thus, the combined system in question, viewed as being under multiple excitations, may be described equally by the discrete equation of motion (15) and consequently admits the solution (16), which can be appropriately written as

$$\boldsymbol{\eta}(x, \omega) = \mathbf{A}(x, x_0, \omega)\boldsymbol{\eta}(x_0, \omega) - \sum_{x_k < x} \mathbf{A}(x, x_k, \omega)\boldsymbol{\eta}^*(x_k, \omega). \tag{17}$$

It can be seen from the above solution that there is no limitation on either the excitations or the response locations, in contrast to the periodic structure solution (16) where both are restricted at points corresponding to integer multiples of the single periodic length. The negative sign on the right-hand side of this solution is due to the assumption that all $\boldsymbol{\eta}^*(x_i, \omega)$, are applied in the positive direction on the left face (negative face) of the repeated continuous subsystems, except the boundary disturbance at the right end, which is assumed to be applied to the right face (positive face) but oriented in a negative direction (Figure 2).

Transfer matrices in (17) can be conveniently diagonalized by wave coordinate transformation of equation (3), by premultiplying both sides by $\mathbf{P}^{-1}(\omega)$, then

$$\begin{Bmatrix} \mathbf{v}^+(x, \omega) \\ \mathbf{v}^-(x, \omega) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^+(x, x_0, \omega) & \\ & \boldsymbol{\Lambda}^-(x, x_0, \omega) \end{bmatrix} \begin{Bmatrix} \mathbf{R}(x_0)\mathbf{v}^-(x_0, \omega) \\ \mathbf{v}^-(x_0. \omega) \end{Bmatrix}$$

$$- \sum_{x_k \leqslant x} \begin{bmatrix} \boldsymbol{\Lambda}^+(x, x_k, \omega) & \\ & \boldsymbol{\Lambda}^-(x, x_k, \omega) \end{bmatrix} [\mathbf{P}]^{-1} \begin{Bmatrix} \mathbf{d}^*(x_k, \omega) \\ \mathbf{f}^*(x_k, \omega) \end{Bmatrix}, \tag{18}$$

where $\mathbf{v}^+(x_0, \omega)$ is replaced by $\mathbf{R}(x_0)\mathbf{v}^-(x_0, \omega)$. $\mathbf{P}^{-1}$ is the inverse matrix of $\mathbf{P}$ and is called the generation matrix since it generates wave components of $\mathbf{v}$ type due to the applied disturbances (or constraints) at their point of application.

The wave solution of (18) may be though of as a sum of two parts: a) a *free wave field*; b) a *forced wave field* generated by discontinuities $\boldsymbol{\eta}^*(x_i, \omega)$ along the (system) portion $x - x_0$ as if the system were infinite in extent. It is apparent that the information regarding the detailed dynamics of the entire system is conveyed through the left boundary incoming wave $\mathbf{v}^-(x_0, \omega)$.

The left boundary incoming wave vector is determined by letting $x = x_m$ in equation (18), where x_m is the system's right end co-ordinate, to give

$$\begin{Bmatrix} \mathbf{v}^+(x_m, \omega) \\ \mathbf{R}(x_m)\mathbf{v}^+(x_m, \omega) \end{Bmatrix} = \begin{bmatrix} \mathbf{\Lambda}^+(x_m, x_0, \omega) & \\ & \mathbf{\Lambda}^-(x_m, x_0, \omega) \end{bmatrix} \begin{Bmatrix} \mathbf{R}(x_0)\mathbf{v}^-(x_0, \omega) \\ \mathbf{v}^-(x_0, \omega) \end{Bmatrix}$$

$$- \sum_{k=0}^{m} \begin{bmatrix} \Lambda^+(x_m, x_k, \omega) & \\ & \mathbf{\Lambda}^-(x_m, x_k, \omega) \end{bmatrix} [\mathbf{P}]^{-1} \begin{Bmatrix} \mathbf{d}^*(x_k, \omega) \\ \mathbf{f}^*(x_k, \omega) \end{Bmatrix}, \tag{19}$$

where $\mathbf{v}^-(x_m, \omega)$ is replaced by $\mathbf{R}(x_m)\mathbf{v}^+(x_m, \omega)$. Then, by premultiplying the first rows of equation (19) by $\mathbf{R}(x_m)$ and subtracting the result from the second rows of the same equation to eliminate the right boundary incoming wave $\mathbf{v}^+(x_m, \omega)$, one has

$$\mathbf{v}^-(x_0, \omega) = \mathbf{\Delta}_0^{-1} \sum_{k \leqslant m} \{\mathbf{\Lambda}^-(x_0, x_m, \omega)\mathbf{R}(x_m)\mathbf{\Lambda}^+(x_m, x_k, \omega)(\mathbf{P}^{-1})_{12}$$

$$- \mathbf{\Lambda}^-(x_0, x_k, \omega)(\mathbf{P}^{-1})_{22}\}\mathbf{f}^*(x_k, \omega), \tag{20}$$

where the matrix $\mathbf{\Delta}_0$ is defined in equation (13).
Notice that, if disturbances are given in terms of imposed deflections (not included in (20) for the sole purpose of simplicity) the positive and negative wave generation matrices $(\mathbf{P}^{-1})_{12}$, $(\mathbf{P}^{-1})_{22}$ must be replaced by $(\mathbf{P}^{-1})_{11}$, $(\mathbf{P}^{-1})_{21}$, respectively.

The physical response can be obtained by use of the transformation equation (3). Thus, the response in terms of deflection variables is given by

$$\mathbf{d}(x, \omega) = [\mathbf{P}_{11}\mathbf{\Lambda}^+(x, x_0, \omega)\mathbf{R}(x_0) + \mathbf{P}_{12}\mathbf{\Lambda}^-(x, x_0, \omega)]\mathbf{v}^-(x_0, \omega)$$

$$- \sum_{x_k \leqslant x} [\mathbf{P}_{11}\mathbf{\Lambda}^+(x, x_k, \omega)(\mathbf{P}^{-1})_{12} + \mathbf{P}_{12}\mathbf{\Lambda}^-(x, x_k, \omega)(\mathbf{P}^{-1})_{22}]\mathbf{f}^*(x_k, \omega). \tag{21}$$

As expected, the continuous system homogeneous solution in (12) and (14) is recovered by (20) and (21), respectively, when all $\mathbf{f}^*(x_k, \omega)$ along the entire system are set to zero.

Equations (20) and (21) give the formal response of the system subjected to point-wise discontinuities. The response in terms of the internal forces at any point along the combined systems is determined by replacing $\mathbf{P}_{11}$, $\mathbf{P}_{12}$ by $\mathbf{P}_{21}$, $\mathbf{P}_{22}$, respectively, in (21) and subtracting the external applied forces at that point (if any) from it. Clearly, the complete solution of combined system dynamics is obtained by imposing the appropriate constraint conditions in this solution, as will be shown through some examples in the next two sections.

Although the response is formulated for constrained/combined systems under pointwise external disturbances, it can easily be modified to accommodate distributed disturbances in terms of applied forces or initial deflections. This can be done by considering the latter as a limiting case of the former. Consequently, this implies that, as the length of the *periodic* subsystem approaches zero, the summation process in (21) and (20) becomes an integration process. Note that the first expression in (21) has no explicit dependency on any type of discontinuity. The integral can be evaluated explicitly and exactly. Indeed, it involves the wave transfer submatrices, which are given in exponential form.

4. APPLICATIONS

4.1. THE MULTI-CONSTRAINED TRANSLATING STRING

Consider a linear uniform string of density ρ and tension T that travels with constant speed $\bar{c}$ between two eyelets separated by a distance L. The string moves across $m-1$ linear elastic springs of stiffness $\bar{s}_i$ located at $\bar{x}_i$, $i = 1, 2, \ldots, m-1$. While the left eyelet remains fixed, the right eyelet is subjected to a prescribed transverse motion of amplitude $\bar{u}^*(L, \omega)$ as shown in Figure 3. The well known non-dimensional equation that governs the homogeneous motion of the unconstrained string is [21]

$$\partial^2 u/\partial t^2 + 2c\,\partial^2 u/\partial x\,\partial t - (1-c^2)\partial^2 u/\partial x^2 = 0, \tag{22}$$

or, equivalently in the frequency domain as

$$\frac{\partial}{\partial x}\begin{Bmatrix} u(x,\omega) \\ \partial u(x,\omega)/\partial x \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2/(1-c^2) & 2jc\omega/(1-c^2) \end{bmatrix}\begin{Bmatrix} u(x,\omega) \\ \partial u(x,\omega)/\partial x \end{Bmatrix}, \tag{23}$$

where $\mathrm{j} = \sqrt{-1}$ and the non-dimensional parameters are defined as

$$x = \bar{x}/L, \qquad u = \bar{u}/L, \qquad s_i = \bar{s}_i L/T, \qquad \omega = \bar{\omega} L\sqrt{\rho/T}, \qquad c^2 = \bar{c}^2/(\rho/T). \tag{24}$$

It follows that the eigenvalues of $\mathbf{F}(\omega)$, the propagation constants of the positive- and negative-going wave components, respectively, are

$$\gamma_{1,2} = (\mathrm{j}\omega/1-c^2)(c \mp 1) \equiv \mathrm{j}(a \mp b), \tag{25}$$

where due to the gyroscopic effect $-\gamma_1 \neq \gamma_2$. Therefore, each wave component travels at a different wave speed; i.e. the positive- and the negative-going wave components travel with speed $1+c$ and $1-c$, respectively.

The mechanical state vector is transformed into positive- and negative-going wave components by the transformation matrix $\mathbf{P}$,

$$\begin{Bmatrix} u(x,\omega) \\ \partial u(x,\omega)/\partial x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{bmatrix}\begin{Bmatrix} v^+(x,\omega) \\ v^-(x,\omega) \end{Bmatrix}. \tag{26}$$

The wave generation matrix is

$$\mathbf{P}^{-1} = \frac{1}{(\gamma_2-\gamma_1)}\begin{bmatrix} \gamma_2 & -1 \\ \gamma_1 & 1 \end{bmatrix}. \tag{27}$$

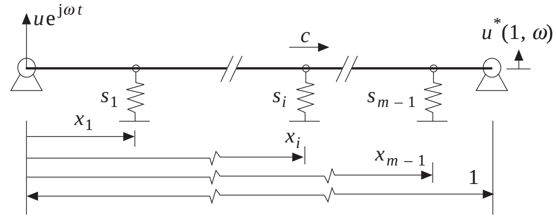

Figure 3. A translating string moving across *m-l* point springs and subjected to a prescribed motion at the right end.

The homogeneous boundary conditions of the string with fixed eyelets may be written in the form

$$[1 \quad 0]\begin{Bmatrix} u(x,\omega) \\ \partial u(x,\omega)/\partial x \end{Bmatrix}_{x=0,1} = 0,$$

which can be transformed into wave components by equation (26), then

$$[1 \quad 1]\begin{Bmatrix} v^{+}(x,\omega) \\ v^{-}(x,\omega) \end{Bmatrix}_{x=0,1} = 0,$$

consequently, the system reflection coefficients are

$$r(x)|_{x=0,1} = -1. \tag{28}$$

Pointwise slope discontinuities exerted by the elastic springs are expressed as

$$\frac{\partial u(x_i,\omega)}{\partial x} = \frac{f^*(x_i,\omega)}{(1-c^2)} = -\frac{s_i}{(1-c^2)}\,u(x_i,\omega), \qquad i = 1,\ldots,m-1. \tag{29}$$

Substituting equations (27)–(29) into (21), the response of the multi-constrained translating string under boundary disturbance $u^*(1,\omega)$ is

$$u(x,\omega) = (e^{\gamma_2 x} - e^{\gamma_1 x})v^{-}(0,\omega) + \sum_{x_k<x}(e^{\gamma_2(x-x_k)} - e^{\gamma_1(x-x_k)})\frac{s_k}{2j\omega}\,u(x_k,\omega), \tag{30}$$

where, according to (20), the incoming wave at the left eyelet is

$$v^{-}(0,\omega) = \frac{1}{(e^{\gamma_1-\gamma_2}-1)}\left[\sum_{k=1}^{m-1}(e^{-\gamma_2\gamma_k} - e^{\gamma_1(1-x_k)-\gamma_2})\frac{s_k}{2j\omega}\,u(x_k,\omega) - e^{-\gamma_2}u^*(1,\omega)\right]. \tag{31}$$

As outlined in section 3, the contribution to the response of the imposed displacement is accounted for by replacing $(\mathbf{P}^{-1})_{12}$ and $(\mathbf{P}^{-1})_{22}$ by $(\mathbf{P}^{-1})_{11}$ and $(\mathbf{P}^{-1})_{21}$ in (19) and (20). Notice that since the imposed deflection is applied at the right eyelet, its contribution appeared only through the left eyelet incoming wave, equation (31).

Equations (30) and (31) suggest two procedures to provide the solution of the forced multi-constrained axially moving string.

4.1.1. *Method* 1.

This method consists of the direct substitution of (31) into (30), which leads to

$$u(x,\omega) = -\frac{\sin bx}{\sin b}\left[\sum_{k=1}^{m-1} e^{j\alpha(x-x_k)}\sin b(1-x_k)\frac{s_k}{\omega}\,u(x_k,\omega) - e^{j\alpha(x-1)}u^*(1,\omega)\right]$$

$$+ \sum_{y_k<x} e^{ja(x-x_k)}\sin b(x-x_k)\frac{s_k}{\omega}\,u(x_k,\omega), \tag{32}$$

where a and b are defined by (25).

A matrix simultaneous equation of order $m-1$ is formed by letting $x \to x_i$, $1, 2, \ldots, m-1$ in (32). This can then be solved numerically to determine the unknown displacements $u(x_i, \omega)$, $i = 1, 2, \ldots, m-1$. For the free vibration case, $u^*(1, \omega) = 0$, the solution of this matrix equation furnishes the constrained system eigenvalues ω_r and eigenvectors $u(x_i, \omega_r)$, $r = 1, 2, 3. \ldots$ The eigenvectors may then be substituted into the general response (32) to determine the system modal shapes. Apart from the relative ease with which this solution is obtained and the simplicity of the solution itself, classical methods would lead to a set of simultaneous equations of $2m$ unknowns (two boundary conditions plus two constraint conditions at each point of the $m-1$ springs location).

Since no explicit solution of this type of problem is available in the literature, consider the free vibration limiting case in which the string moves across a single spring of stiffness s_1 located at point x_1. Then, from solution (32), it is clear that the characteristic equation is

$$\sin b + (s_1/\omega) \sin b(1 - x_1) \sin bx_1 = 0, \tag{33}$$

which is identical to that given in [22].

4.1.2. *Method* 2.

The second method consists of implementing the phase closure principle stated in section 2. This is accomplished by expressing the *forced field* (the second term on the right side of (30)) through the boundary incoming wave $v^-(0, \omega)$. The physical implication of this is to account for the multiple reflections of $v^-(0, \omega)$ encountering the intermediate constraints before completing its *global round-trip* path and returning to the starting position. Due to the readily included continuity conditions in equation (30), the contribution of the *forced field* constraint forces, $s_k u(x_k, \omega)$, to the displacement at each constraint location is composed of those positioned to the left of this point. Therefore, it is evident that all $s_k u(x_k, \omega)$ in equation (30) are expressible in terms of $v^-(0, \omega)$ by repeated substitution of the preceding point displacements. To be more specific, let one consider the case of the string that is vibrating freely (with frequency ω) and passes over three point springs. By (30) the displacement at x_1 is

$$u(x_1, \omega) = (e^{\gamma_2 x_1} - e^{\gamma_1 x_1}) v^-(0, \omega) = 2j\, e^{j\alpha x_1} \sin bx_1 v^-(0, \omega). \tag{34}$$

The displacement at x_2 is given by

$$\begin{aligned} u(x_2, \omega) &= (e^{\gamma_2 x_2} - e^{\gamma_1 x_2}) v^-(0, \omega) + [(e^{\gamma_2 (x_2 - x_1)} - e^{\gamma_1 (x_2 - x_1)})/2j\omega] s_1 u(x_1, \omega) \\ &= 2j\, e^{jax_2} [\sin bx_2 + (s_1/\omega) \sin bx_1 \sin b(x_2 - x_1)] v^-(0, \omega), \end{aligned} \tag{35}$$

where the $u(x_1, \omega)$ is substituted by the expression given in (34).

Similarly, the displacement at x_3:

$$\begin{aligned} u(x_3, \omega) &= (e^{\gamma_2 x_3} - e^{\gamma_1 x_3}) v^-(0, \omega) + \sum_{k=1}^{2} \frac{(e^{\gamma_2 (x_2 - x_k)} - e^{\gamma_1 (x_2 - x_k)})}{2j\omega} s_k u(x_k, \omega) \\ &= 2j\, e^{jax_3} \{\sin bx_3 + (s_1/\omega) \sin bx_1 \sin b(x_3 - x_1) \\ &\quad + (s_2/\omega) \sin b(x_3 - x_2)[\sin bx_2 + (s_1/\omega) \sin bx_1 \sin b(x_2 - x_1)]\} v^-(0, \omega), \end{aligned} \tag{36}$$

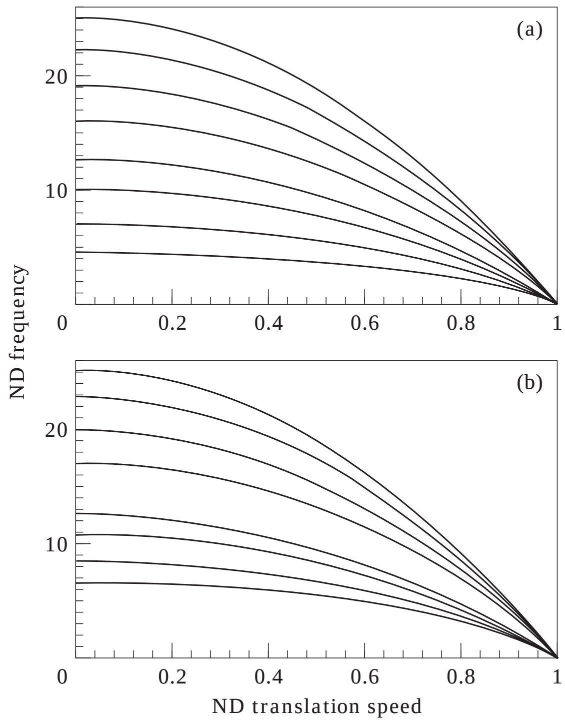

Figure 4. The first eight natural frequencies for a fixed–fixed translating string with three equispaced linear springs having equal ND stiffness (a) s = 2·5 and (b) s = 10.

where $u(x_1, \omega)$ and $u(x_2, \omega)$ are replaced by equations (34) and (35), respectively. By inserting (34)–(36) into (31), on finds

$$\Delta_c/\Delta_0 v^-(x_0, \omega) = 0, \tag{37}$$

where the system poles ($\Delta_0 = 0$) correspond to the natural frequencies of the classical threadline [21]. According to the phase closure principle, the natural frequencies of the constrained translating string satisfy:

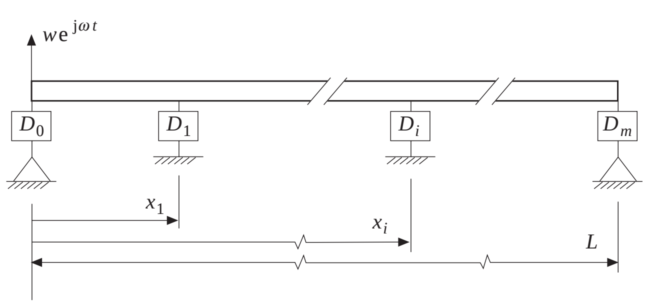

Figure 5. Schematic of a beam system with simple end supports and interior constraint discontinuities D_i; D_0 and D_m represent the system inhomogeneous boundary conditions.

TABLE 1

Eigenvalues (non-dimensional wave number μL) of a six-span beam on equispaced translational spring supports with simply supported ends (non-dimensional (ND) stiffness $s = \bar{s}L^3/EI$)

ND stiffness parameter ($s = 2{\cdot}16$)		ND stiffness parameter ($s = 216$)		ND stiffness parameter ($s = 2{\cdot}16 \times 10^6$)	
Present method	Ref. [5]	Present method	Ref. [5]	Present method	Ref. [5]
3·2412	3·241	6·1076	6·108	18·8496	6π
6·2962	6·296	7·3080	7·308	19·5536	19·554
9·4286	9·429	9·7892	9·789	21·3037	21·304
12·5680	12·568	12·7259	12·725	23·4866	23·487
15·7088	15·709	15·7903	15·790	25·6568	25·657
18·8496	6π	18·8496	6π	27·4084	27·408
21·9915	21·991	22·0218	22·022	37·6991	12π
25·1329	25·133	25·1532	25·153	38·3872	38·387

$$\begin{aligned}\Delta_c = & -\sin b - (s_1/\omega)\sin bx_1 \sin b(1-x_1) - (s_3/\omega)\sin b(1-x_3) \\ & \times [\sin bx_3 + (s_1/\omega)\sin bx_1 \sin b(x_3 - x_1)] \\ & - (s_2/\omega)[\sin bx_2 + (s_1/\omega)\sin bx_1 \sin b(x_2 - x_1)] \\ & \times [\sin b(1-x_2) + (s_3/\omega)\sin b(1-x_3)\sin b(x_3 - x_2)] = 0. \end{aligned} \tag{38}$$

Of course the same frequency equation as (38) could be obtained using method 1 and in this case no computational advantage should be attributed to the phase closure in finding the natural frequencies, even though the use of phase closure appeared to reduce the characteristic equation to the order of the cross-sectional negative wave components. However, some superiority may be noticed in finding the system mode shapes. This is because all system complex mode shapes can be given in terms of $v^-(0, \omega)$ by inserting equations (34)–(36) into (30), without the need to find the eigenvectors $u(x_1, \omega_r)$ for each frequency as required by method 1.

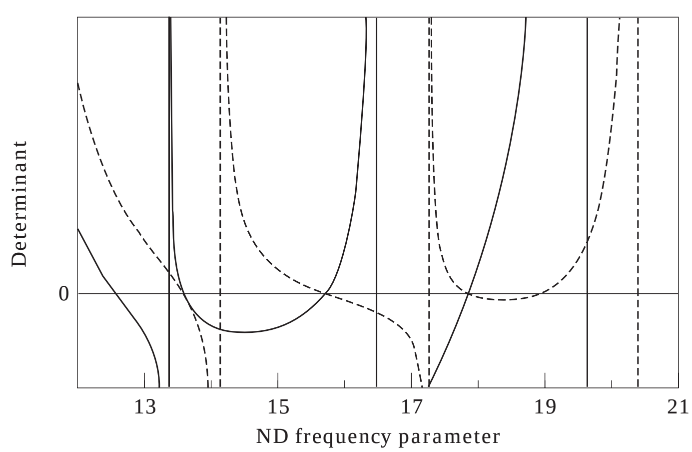

Figure 6. Graphical solution to eigenvalue equation of a four equal span beam on simple interior supports: with simply supported ends (solid curves) and clamped ends (dashed curves).

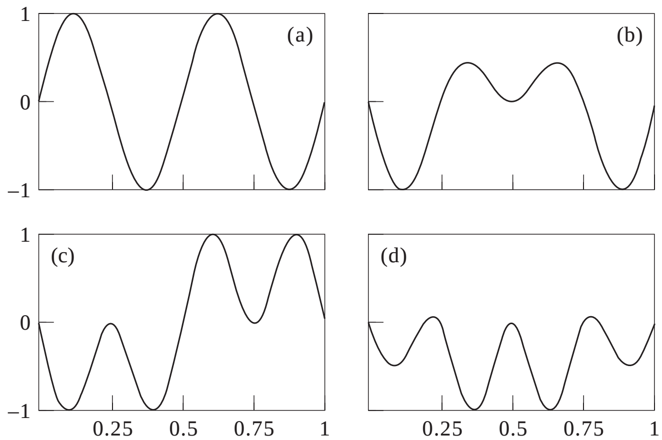

Figure 7. Mode shapes of a four equal span simply supported beam: (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4.

If the system is subjected to a boundary motion at the right eyelet, from (31) and (37) it is clear that

$$v^{-}(x_0, \omega) = \frac{e^{-\gamma_2} u^*(1, \omega)}{\Delta_c} \tag{39}$$

The forced mode shapes are determined by the same procedure that gives the free modes except that here the incoming wave is uniquely defined by (39).

In order to verify this method, let us examine the following especial cases. For the free vibration limiting case $s_1 = s_2 = s_3 \to \infty$, the characteristic equation (38) reduces to

$$\sin bx_1 \sin b(x_2 - x_1) \sin b(x_3 - x_2) \sin b(1 - x_3) = 0, \tag{40}$$

TABLE 2

Eigenvalues (non-dimensional wave number μL) of a equispaced-four-span beam on simple interior supports

Simply supported ends		Clamped ends		Clamped–simply supported ends	
Present method	Periodic structures theory	Present method	Periodic structures theory	Present method	Periodic structures theory
12·5664	12·56	13·5729	13·57	12·8403	12·84
13·5729	13·57	15·7064	15·71	14·5816	14·58
15·7064	15·71	17·8533	17·85	16·8322	16·83
17·8533	17·85	18·9202	18·92	18·6210	18·62
25·1327	25·13	26·1817	26·18	25·4276	25·43
26·1817	26·18	28·2743	28·27	27·1795	27·18
28·2743	28·27	30·3665	30·37	29·3691	29·37
30·3665	30·37	31·4128	31·41	31·1191	31·12

which provides four sets of natural frequencies corresponding to four decoupled classical threadlines of length x_1, $x_2 - x_1$, $x_3 - x_2$ and $1 - x_3$, respectively. If $s_2 \to \infty$, then the characteristic equation (38) becomes

$$[\sin b(1 - x_2) + (s_3/\omega)\sin b(1 - x_3)\sin b(x_3 - x_2)][\sin bx_2 + (s_1/\omega)\sin bx_1 \sin b(x_2 - x_1)] = 0, \quad (41)$$

which yields two sets of natural frequencies corresponding to two decoupled string systems of length x_2 and $1 - x_2$ constrained by springs of stiffness s_1 and s_2. As expected, each set in this solution agrees with that obtained by method 1 in (33).

A numerical solution of frequency equation (38) is performed for the case with three equal-equispaced springs of non-dimensional (ND) stiffness s. It is found that the system natural frequencies tend to occur in groups (four natural frequencies in each group). The first two groups of natural frequencies are plotted in Figure 4 as a function of the translation speed and spring stiffness, for $s = 2{\cdot}5$ (Figure 4a) and $s = 10$ (Figure 4b). The spacing between these groups, for fixed translation speed c, is proportional to the spring stiffness s. This is because the upper natural frequencies of each group coincide with the natural frequencies of the classical threadline of length 0·25 and hence are independent of the value of the spring stiffness, while the lower frequencies increase with s. For fixed s, the system natural frequencies decrease monotonically with translation speed c and vanish when this speed approaches unity. Hence, the spring constraints do not influence the string's critical speed, in agreement with [22].

4.2. BEAMS ON MULTIPLE SUPPORTS

Figure 5 shows a schematic diagram of an Euler–Bernoulli beam system of total length $x_m - x_0 = L$ with simply supported ends. The beam is connected to $m - 1$ pointwise support discontinuities D_i at sections x_i, $i = 1, 2, \ldots, m - 1$. Each support discontinuity may generate a shear force $V^*(x_i, \omega)$ and/or a bending moment $M^*(x_i, \omega)$. Additionally, boundary disturbances in terms of $M^*(x_i, \omega)$ and $w^*(x_i, \omega)$, $i = 0, m$, are introduced to include other end conditions in the solution without the need to modify the system reflection matrices.

The partial differential equation governing the free flexural motion of the undamped beam is

$$EI\, \partial^4 w/\partial x^4 + \rho A\, \partial^2 w/\partial t^2 = 0 \quad (42)$$

where EI is the beam flexural stiffness, ρ is the mass density and A is the cross-sectional area of the beam. For a given frequency ω, the equivalent state space representation of equation (42) is

$$\frac{\partial}{\partial x}\begin{Bmatrix} w(x,\omega) \\ \varphi(x,\omega) \\ M(x,\omega)/EI \\ -V(x,\omega)/EI \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 A\rho/EI & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w(x,\omega) \\ \varphi(x,\omega) \\ M(x,\omega)/EI \\ -V(x,\omega)/EI \end{Bmatrix}, \quad (43)$$

where w, φ, M and V are the transverse displacement, slope, bending moment and shear force of the beam, respectively. The eigenvalues of the matrix in equation (43) are

$$\gamma_{1,3} = \mp \mathrm{j}\mu, \qquad \gamma_{2,4} = \mp \mu, \qquad \mu = \sqrt[4]{A\rho\omega^2/EI}, \quad (44)$$

where $\gamma_1, \gamma_2(\gamma_3, \gamma_4)$ are the frequency dependent propagation constants of positive-going (negative-going) travelling and evanescent wave components.

The cross-sectional mechanical state of the beam transformed into wave state as follows

$$\begin{Bmatrix} w(x,\omega) \\ \varphi(x,\omega) \\ M(x,\omega)/EI \\ -V(x,\omega)/EI \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\mathrm{j}\mu & -\mu & \mathrm{j}\mu & \mu \\ -\mu^2 & \mu^2 & -\mu^2 & \mu^2 \\ \mathrm{j}\mu^3 & -\mu^3 & -\mathrm{j}\mu^3 & \mu^3 \end{bmatrix} \begin{Bmatrix} v_t^+(x,\omega) \\ v_e^+(x,\omega) \\ v_t^-(x,\omega) \\ v_e^-(x,\omega) \end{Bmatrix}, \tag{45}$$

where subscripts t and e denote travelling and evanescent wave components. The wave generation matrix is

$$\mathbf{P}^{-1}(\omega) = \frac{1}{4}\begin{bmatrix} 1 & \mathrm{j}/\mu & -1/\mu^2 & -\mathrm{j}/\mu^3 \\ 1 & -1/\mu & 1/\mu^2 & -1/\mu^3 \\ 1 & -\mathrm{j}/\mu & -1/\mu^2 & \mathrm{j}/\mu^3 \\ 1 & 1/\mu & 1/\mu^2 & 1/\mu^3 \end{bmatrix}, \tag{46}$$

The system homogeneous boundary conditions,

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Bmatrix} \begin{Bmatrix} w(x_i,\omega) \\ \varphi(x_i,\omega) \\ M(x_i,\omega)/EI \\ -V(x_i,\omega)/EI \end{Bmatrix}_{i=0,m} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{47}$$

are transformed into wave state

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -\mu^2 & \mu^2 & -\mu^2 & \mu^2 \end{bmatrix} \begin{Bmatrix} \mathbf{v}^+(x_i,\omega) \\ \mathbf{v}^-(x_i,\omega) \end{Bmatrix}_{i=0,m} = 0. \tag{48}$$

By comparing (7) and (9) with the above equations, the beam reflection matrices are given by

$$\mathbf{R}(x_i)|_{i=0,m} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{49}$$

According to the wave transfer matrix method described in section 3, the beam system's formal wave response is determined by equation (18) and consequently the system mechanical states can be obtained by using the transformation equation (3). As the procedure for doing this is straightforward, the response in terms of the beam's cross-sectional entire mechanical state will not be given. The main purpose of this example is to show how the free response of the beam on multiple supports under arbitrary boundary conditions can be determined in terms of simpler boundary conditions, as the case of simple end supports. This procedure is described in detail for beams with simple–free ends and clamped–simple ends, while for the clamped–clamped beam the results are given directly in Appendix A. Other types of boundaries or constraints may likewise be included without difficulties. The beam system's vertical displacement is given as

$$w(x, \omega) = \left\{\frac{\sin \mu x \cos \mu L}{\sin \mu L} - \frac{\sinh \mu x \cosh \mu L}{\sinh \mu L} + (\cosh \mu x - \cos \mu x)\right\} \frac{M^*(x_0, \omega)}{2\mu^2 EI}$$

$$- \left\{\frac{\sin \mu x}{\sin \mu L} - \frac{\sinh \mu x}{\sinh \mu L}\right\} \frac{M^*(x_m, \omega)}{2\mu^2 EI} + \left\{\frac{\sin \mu x}{\sin \mu L} + \frac{\sinh \mu x}{\sinh \mu L}\right\} \frac{w^*(x_m, \omega)}{2}$$

$$- \left\{\frac{\sin \mu x \cos \mu L}{\sin \mu L} + \frac{\sinh \mu x \cosh \mu L}{\sinh \mu L} - (\cosh \mu x + \cos \mu x)\right\} \frac{w^*(x_0, \omega)}{2}$$

$$+ \sum_{k=1}^{m-1} \left\{\frac{\sin \mu x \sin \mu(L - x_k)}{\sin \mu L} - \frac{\sinh \mu x \sinh \mu(L - x_k)}{\sinh \mu L}\right\} \frac{V^*(x_k, \omega)}{2\mu^3 EI}$$

$$+ \sum_{x_k < x} \{\sinh \mu(x - x_k) - \sin \mu(x - x_k)\} \frac{V^*(x_k, \omega)}{2\mu^3 EI} . \tag{50}$$

where, for brevity, interior constraints in terms of $M^*(x_i, \omega)$ are not included. On the basis of this solution, let one consider the following end condition cases.

4.2.1. *Simply–supported ends*

As this case corresponds to a beam system with homogeneous boundary conditions, all boundary disturbances are set to zero, leading to

$$w(x, \omega) = \sum_{k=1}^{m-1} \left\{\frac{\sin \mu x \sin \mu(L - x_k)}{\sin \mu L} - \frac{\sinh \mu x \sinh \mu(L - x_k)}{\sinh \mu L}\right\} \frac{V^*(x_k, \omega)}{2\mu^3 EI}$$

$$+ \sum_{x_k < x} \{\sinh \mu(x - x_k) - \sin \mu(x - x_k)\} \frac{V^*(x_k, \omega)}{2\mu^3 EI} . \tag{51}$$

4.2.2. *Clamped–simply supported ends*

The left boundary condition for this case is the vanishing of $\varphi(x_0, \omega)$. Since the right boundary remains homogeneous ($M^*(x_m, \omega) = 0$), the disappearance of the slope at x_0 enables one to write the unknown reaction moment at that location in terms of all unknown reaction forces along the beam. This is given by

$$\frac{M^*(0, \omega)}{2EI} = -\frac{1}{\Delta_{C-S}} \sum_{k=1}^{m} \{\sin \mu(L - x_k) \sinh \mu L - \sinh \mu(L - x_k) \sin \mu L\} \frac{V^*(x_k, \omega)}{2\mu EI} , \tag{52}$$

where $\Delta_{C-S} = \cos \mu L \sinh \mu L - \cosh \mu L \sin \mu L$.

By substituting the above equation in (50), the system's transverse deflection for these boundary conditions is

$$w(x, \omega) = \frac{1}{\Delta_{C-S}} \Bigg\{ -(\cosh \mu x - \cos \mu x) \sum_{k=1}^{m} [\sin \mu(L - x_k) \sinh \mu L - \sinh \mu(L - x_k) \sin \mu L] + (\sinh \mu x - \sin \mu x) \sum_{k=1}^{m} [\sin \mu(L - x_k) \cosh \mu L - \sinh \mu(L - x_k) \cos \mu L] \Bigg\} \frac{V^*(x_k, \omega)}{2\mu^3 EI} + \sum_{x_k < x} \sinh \mu(x - x_k) - \sin \mu(x - x_k)] \frac{V^*(x_k, \omega)}{2\mu^3 EI}. \quad (53)$$

4.2.3. *Simply supported-free ends*

For this case, the *additional* boundary condition is the disappearance of the shear force $V(x_m, \omega)$ due to the imposed deflection $w^*(x_m, \omega)$. This leads to

$$\frac{w^*(x_m, \omega)}{2} = +\frac{1}{\Delta_{S-F}} \sin \mu L \sinh \mu L \sum_{x_k < x} [\cosh \mu(L - x_k) + \cos \mu(L - x_k)] \frac{V^*(x_k, \omega)}{2\mu^3 EI} - \frac{1}{\Delta_{S-F}} \sum_{k=1}^{m-1} \{\cos \mu L \sin \mu(L - x_k) \sinh \mu L + \cosh \mu L \sinh \mu(L - x_k) \sin \mu L\} \frac{V^*(x_k, \omega)}{2\mu^3 EI}, \quad (54)$$

where $\Delta_{S-F} = \cos \mu L \sinh \mu L - \cosh \mu L \sin \mu L$.

By inserting (54) into (50), the displacement for this case is

$$w(x, \omega) = \frac{1}{\Delta_{S-F}} \Bigg\{ (\sinh \mu x \sin \mu L + \sin \mu x \sinh \mu L) \sum_{k=1}^{m} [\cosh \mu(L - x_k) + \cos \mu(L - x_k)] - (\sinh \mu x \cos \mu L + \sin \mu x \cosh \mu L) \sum_{k=1}^{m} [\sinh \mu(L - x_k) + \sin \mu(L - x_k)] \Bigg\} \times \frac{V^*(x_k, \omega)}{2\mu^3 EI} + \sum_{x_k < x} [\sinh \mu(x - x_k) - \sin \mu(x - x_k)] \frac{V^*(x_k, \omega)}{2\mu^3 EI}. \quad (55)$$

Numerical verifications of these solutions are conducted and compared with those appearing in the literature; namely, a beam system with flexible intermediate supports and simply supported ends. The matrix characteristic equation of the combined system is constructed by letting $w(x, \omega) \rightarrow w(x_i, \omega)$ and inserting constraint conditions $V^*(x_i, \omega) = -\bar{s}_i w(x_i, \omega)$ in (51). The special case of a six-span beam system on equispaced transverse spring supports all of stiffness $\bar{s}$ is solved numerically. This system's natural frequencies (in terms of the non-dimensional wave number μL) are given in Table 1 and compared with those given in reference [5]. It is clear that these results are in agreement.

Another example consists of a four-equal-span beam system on simple intermediate supports; $w(x_i, \omega) = 0$, $i = 1, 2, 3$; and with: simply supported ends, clamped–simply supported ends and clamped–clamped ends is also considered. The first four natural frequencies of the beam system with simply supported ends (solid curves) and clamped ends (dashed curves) are evaluated graphically in Figure 6. Note that for the simply supported ends case (solid curves in Figure 6) the system's homogeneous boundary conditions are taken as simple–free end supports in order to capture the degenerate mode (the constrained system natural frequency (which coincides with that of the unconstrained one; the first natural frequency in Figure 6 for the present case). This frequency, otherwise, could be missed if the homogeneous boundary conditions are considered as simple end supports as discussed by Bergman and McFarland [5]. The corresponding mode shapes of these frequencies are plotted in Figure 7. Table 2 contains the natural frequencies of this beam system with different end conditions. These frequencies are compared with those obtained by periodic structures theory using Sen Gupta's graphical method [23]. Again, both methods are in agreement.

5. CONCLUSIONS

It has been shown that frequency domain dynamic analysis of combined systems can be performed in an exact closed form and a unified manner by adopting multiexcited periodic waveguide solution. It is found that analyzing constrained/combined systems by the method described in this paper has advantages over the conventional analytical methods:

The acquired unity due to the consideration of the system as a whole (as a multi-excited periodic structure) has the merit of reducing the computational efforts, since one needs only to enforce constraint conditions along the combined system while displacements continuity at all constraints locations are automatically taken into account. Note that conventional methods require the enforcement of both continuity and force balance at each point of constraint discontinuity.

The discrete formulation of the solution enables one to treat different problems in a unified and compact manner. It does not matter if the system is stationary or gyroscopic subjected to boundary or interior disturbances and constraints: the formulation of the solution remains basically the same.

The spatial description due to the introduced wave propagation viewpoint gives a physical insight into the dynamics of combined systems. Moreover, a further reduction in the computational efforts is gained, because the solution derived by this method is given in terms of wave vectors through the formulation of the wave transfer matrix, which is diagonal, rather than in terms of the mechanical vector through the formulation of the fully populated mechanical transfer matrix (a procedure used in the conventional transfer matrix method and the transfer function method).

Two examples demonstrate the proposed approach, which can be conveniently used in structural vibration power flow analysis and wave control.

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APPENDIX A

Clamped–Clamped Beam System

The transverse displacement of a clamped-clamped beam system under multiple constraint forces $V^*(x_i, \omega)$ is

$$w(x, \omega) =$$

$$\frac{1}{\Delta_{C-C}}\Bigg[(\sinh \mu x - \sin \mu x)\sum_{k-1}^{m-1}\{(\sinh \mu L + \sin \mu L)[\sinh\ \mu(L - x_k) - \sin \mu(L - x_k)]$$

$$- (\cosh \mu L - \cos \mu L)[\cosh \mu(L - x_k) - \cos \mu(L - x_k)]\}$$

$$+ (\cosh \mu x - \cos \mu x) \sum_{k-1}^{m-1} \{(\sinh \mu L - \sin \mu L)[\cosh \ \mu(L - x_k) - \cos \mu(L - x_k)]$$

$$- (\cosh \mu L - \cos \mu L)[\sinh \mu(L - x_k) - \sin \mu(L - x_k)]\}\Bigg] \frac{V^*(x_k, \omega)}{2\mu^3 EI}$$

$$+ \sum_{x_k < x} \sinh \mu(x - x_k) - \sin \mu(x - x_k)] \frac{V^*(x_k, \omega)}{2\mu^3 EI}$$

where $\Delta_{C-C} = 2(1 - \cos \mu L \cosh \mu L)$.